Journal of Sound and Vibration 325 (2009) 781–797

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Vibrations of porous piezoelectric ceramic plates

Anil K. Vashishth *, Vishakha Gupta

Department of Mathematics, Kurukshetra University, Kurukshetra 136119, India

ARTICLE INFO

Article history:
Received 17 June 2008
Received in revised form
16 March 2009
Accepted 21 March 2009
Handling Editor: L.G. Tham
Available online 29 April 2009

ABSTRACT

The basic constitutive equations and equations of motion are derived for anisotropic porous piezoelectric materials by making use of variational principle. These equations are first derived for three-dimensional case and then reduced to the two-dimensional case by expanding mechanical displacement and electric potential as power series. The results are obtained in particular for monoclinic (2) porous piezoelectric materials. Thickness shear correction factors are determined for high frequency case. The thickness shear resonance frequencies are obtained numerically for a particular model of PZT.

0. Introduction

It is well known that piezoelectric materials produce an electric field when deformed and undergo deformation when subjected to an electric field. Piezoelectric materials are integrated with structural systems to form a class of smart structures. Piezoelectric materials have wide-spread applications in many areas such as electronic technology, mechanical engineering, medical appliance and other modern industrial fields. During few recent decades, piezoelectric materials have been intensively used as transducers, actuators, sensors etc. Piezoelectric materials and components are always fabricated in plates or shell configuration for engineering use. The constitutive equations for piezoelectric crystals of different kinds of anisotropy were described by Cady [1]. The characteristics of different piezoelectric materials are described in the text Mason [2]. These equations are also described in the text Holland and Nisse [3]. Based on an approximation involving the early terms of the thickness coordinate of plate, the classical equations of piezoelectricity were reduced from three to two dimensions by Mindlin [4]. Only flexure, thickness-shear and thickness-twist modes were taken into account. Later on, the equations were extended to include the face-extension and face-shear modes by Tiersten and Mindlin [5]. A detailed text on vibrations of linear piezoelectric plates was presented by Tiersten [6]. The two-dimensional equations of motion were derived by Mindlin [7] using expansion of mechanical displacement and electrical potential as power series. The thickness-shear correction factors were also determined.

Shear horizontal vibration modes of plates are often used for bulk acoustic wave piezoelectric resonators and other devices [8]. Shaw [9] discovered the edge modes in piezoelectric finite cylindrical disks with edge conditions in addition to the boundary conditions. He investigated the thickness, shear and radial vibrations of thick barium titanate disk by optical interference methods in wide range of thickness/radius ratio. Some studies [10–13] on thickness-shear and flexure vibrations of plates have also been made by different authors. A new theory for electroded piezoelectric plates and its finite element applications for the forced vibrations of quartz crystal resonators were presented [14]. The general vibration problem of a spherically isotropic piezoelectric medium with radial inhomogeneity was studied by Chen [15] using separation method. The nature of coupling field was investigated for some cases of longitudinal waves propagating in thin, infinitely long, piezoceramics rods with their axes parallel to poling axis [16]. Piquette [17] obtained the coupling coefficients for electrostrictive ceramics. Yang [18] obtained an exact solution for shear horizontal vibrations of a

* Corresponding author.
E-mail address: anil_vashishth@yahoo.co.in (A.K. Vashishth).

0022-460X/$ - see front matter © 2009 Published by Elsevier Ltd.
doi:10.1016/j.jsv.2009.03.034

piezoelectric wedge of polarized ceramics. The steady-state vibrations of an infinite piezoelectric medium with transversely isotropic symmetry were studied by Lioubimova and Schiavone [19], by considering fundamental boundary value problems in a theory of generalized plane strain.

Free vibration occurs when a mechanical system is set off with an initial input and then allowed to vibrate freely. A uniformizing method [20] was presented for free vibration analysis of metal–piezoceramics composite thin plates with different shapes, vibration modes and boundary conditions. In Refs. [21,22], the free vibrations of piezoelectric circular plates, by employing the general solution for coupled three-dimensional equations of transversely isotropic piezoelectric body were investigated. Based on newly presented state space formulations, a method [23] was developed for analyzing the bending, vibrations and stability of laminated transversely isotropic rectangular plates with simply supported edges. An analytical solution was obtained for free vibrations of piezoelectric annular plates by using general solutions for coupled three-dimensional equations of piezoelectric media [24]. The resonant electromechanical vibrations of thin piezoelectric plates under elastic loading were studied by Karlash [25].

The concepts of functionally grade materials (FGM) have been applied to piezoelectric materials to improve its lifetime and reliability of advanced piezoelectric structures. Different authors [26,27] have addressed the problems related to the free vibrations of such functionally grade materials.

Piezoelectric layers are embedded in or surface bonded on isotropic or laminated composite structures to form (smart) structures. Such structures are defined as "piezoelectric composite laminates". Vibration characteristics of piezoelectric composite materials are of interest in the field of Science and Engineering. A lot of work has been done in the field. Mention a few [28–35], in which number of problems related to vibrations of laminated piezoelectric composite materials, have been investigated.

A set of equations of high frequency vibrations of piezothermoelastic crystal plate was obtained by Mindlin [36]. Three-dimensional solutions for free vibrations of initially stressed thermoelectroelastic multilayered plates have been obtained in Ref. [37]. The nonlinear vibration behavior of piezoelectric materials was observed in Refs. [38–40].

Despite the significant progress made in enhancing the coupling characteristics between electrical and mechanical properties in piezoelectric materials, monolithic piezoelectric materials generally exhibit limitation such as brittleness. Due to brittleness nature of piezoelectric ceramics and possible defects of impurity, cavities and microcracks, failure of devices take place easily under mechanical or electrical loading. In order to overcome this limitation, material density is reduced through the addition of controlled porosity and resulting porous piezoelectric materials (PPM) are widely used for applications such as low frequency hydrophones, miniature, accelerometers, vibratory sensors and contact microphones. Due to lower acoustic impedance, these materials can be used to improve mismatch of acoustic impedances at the interfaces of medical ultrasonic imaging devices or underwater sonar detectors. Several piezoceramic constructive elements are porous especially when they are hot pressured or cast under pressure. Use of the piezoelectric effect in porous piezoelectric ceramics offers an original method for studying the coupling between electrical, mechanical, permeability and of course piezoelectric properties of porous systems. Khoroshun et al. [41,42] presented a general approach, to calculate the effective electroelastic properties of polycrystals with spheroidal crystallites, based on conditional averaging of the electroelasticity equations. Khoroshun and Dorodnykh [43,44] evaluated the effective electroelastic properties of porous polycrystals with trigonal symmetry and observed that all the effective constants depend considerably on both the concentration and shapes of the pores. Similar results were also obtained by Gupta and Venkatesh [45] by using a finite element model for porous piezoelectric ceramics. A survey of literature reveals that while a lot of work has been done on vibration analysis of piezoelectric material, piezothermoelastic materials and other smart materials, but no work has been done on vibration analysis of porous piezoelectric materials so far.

In this paper, an attempt is made to formulate the equations of the vibration of unclamped porous piezoelectric crystal plates. In Section 1, using Biot's theory and electrical enthalpy density function, we first derive basic constitutive equations for anisotropic porous piezoelectric materials and then equations of motion for PPM are derived using variational principle. In Section 2, the constitutive relations and equation of motion are obtained for two-dimensional case by expanding mechanical displacement and electrical potential function as power series. In Section 3, particular cases of constitutive relations are obtained with approximation of series after truncation and adjustment of some terms. Further expressions are modified by introducing thickness-shear correction factors. The results are obtained in particular for high frequency vibrations. The thickness-shear correction factors are determined for the Monoclinic (2) crystal by comparing thickness-shear frequencies corresponding to the two- and three-dimensional case in Section 4. Finally, in Section 5, the thickness shear resonance frequencies are obtained numerically for a particular model of PZT and variation of resonance frequency in thickness shear mode with the thickness of the plate is observed. The effect of thickness of plate on the three-dimensional thickness shear resonance frequency for first five modes is also observed.

1. Three-dimensional equation

1.1. Constitutive equations

Let us consider a porous piezoelectric body having a volume V bounded by a surface S with unit outward normal $\hat{n}$. Let $\sigma_{ij}, \varepsilon_{ij}; \sigma^*, \varepsilon^*$ be stress and strain components for solids; fluid phase of porous aggregate, respectively. The quantities with * are associated to the fluid component of porous bulk material. Let $E_i, D_i; E_i^*, D_i^*$ be electric field and electric displacement of solid; fluid phase, respectively.

The electric enthalpy density function (W) for porous piezoelectric material is defined as

$$W = \tfrac{1}{2}[\sigma_{ij}\varepsilon_{ij} + \sigma^*\varepsilon^* - E_iD_i - E_i^*D_i^*] \tag{1}$$

This enthalpy density function W is a quadratic function of $\varepsilon_{ij}, \varepsilon^*, E_i$ and E_i^*.

$$W = \tfrac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \tfrac{1}{2}R\varepsilon^*\varepsilon^* + m_{ij}\varepsilon_{ij}\varepsilon^* - e_{kij}E_k\varepsilon_{ij} - \zeta_{kij}E_k^*\varepsilon_{ij} - \tilde{\zeta}_iE_i\varepsilon^* - e_i^*E_i^*\varepsilon^* - \tfrac{1}{2}\xi_{ij}E_iE_j - \tfrac{1}{2}\xi_{ij}^*E_i^*E_j^* - A_{ij}E_iE_j^*, \tag{2}$$

where coefficients $c_{ijkl}; e_{kij}, \zeta_{kij}; m_{ij}, \xi_{ij}, \xi_{ij}^*, A_{ij}; e_i^*, \tilde{\zeta}_i$ and R are tensors of order 4; 3; 2; 1 and zero, respectively.

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \varepsilon^* = \boldsymbol{U}_{k,k}^*, \quad E_i = -\phi_{,i}, \quad E_i^* = -\phi_{,i}^*, \tag{3}$$

where $u_i, \phi; U_i^*, \phi^*$ are mechanical displacements and electric potential for solid; fluid phase, respectively.

Eq. (2) gives us

$$\frac{\partial W}{\partial \varepsilon_{ij}} = c_{ijkl}\varepsilon_{kl} + m_{ij}\varepsilon^* - e_{kij}E_k - \zeta_{kij}E_k^*, \tag{4a}$$

$$\frac{\partial W}{\partial \varepsilon^*} = R\varepsilon^* + m_{ij}\varepsilon_{ij} - \tilde{\zeta}_iE_i - e_i^*E_i^*, \tag{4b}$$

$$\frac{\partial W}{\partial E_k} = -e_{kij}\varepsilon_{ij} - \tilde{\zeta}_k\varepsilon^* - \xi_{kj}E_j - A_{kj}E_j^*, \tag{4c}$$

$$\frac{\partial W}{\partial E_k^*} = -\zeta_{kij}\varepsilon_{ij} - e_k^*\varepsilon^* - A_{jk}E_j - \xi_{kj}^*E_j^*. \tag{4d}$$

It is known from the definition of the electric enthalpy density function that

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij}, \quad \frac{\partial W}{\partial \varepsilon^*} = \sigma^*, \quad \frac{\partial W}{\partial E_i} = -D_i, \quad \frac{\partial W}{\partial E_i^*} = -D_i^*. \tag{5}$$

This implies that

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} + m_{ij}\varepsilon^* - e_{kij}E_k - \zeta_{kij}E_k^*, \tag{6a}$$

$$\sigma^* = m_{ij}\varepsilon_{ij} + R\varepsilon^* - \tilde{\zeta}_iE_i - e_i^*E_i^*, \tag{6b}$$

$$D_i = e_{ijk}\varepsilon_{jk} + \tilde{\zeta}_i\varepsilon^* + \xi_{ij}E_j + A_{ij}E_j^*, \tag{6c}$$

$$D_i^* = \zeta_{ijk}\varepsilon_{jk} + e_i^*\varepsilon^* + A_{ij}E_j + \xi_{ij}^*E_j^*. \tag{6d}$$

These are constitutive equations for anisotropic porous piezoelectric materials. Here c_{ijkl} are elastic stiffness constants. The elastic constant R measures the pressure to be exerted on fluid to push its unit volume into the porous matrix. $e_{kij}, \xi_{ij}; e_i^*, \xi_{ij}^*$ are piezoelectric and dielectric constant for solid; fluid phase, respectively. $m_{ij}; \zeta_{kij}, \tilde{\zeta}_i; A_{ij}$ are the parameters which take into account the elastic; piezoelectric; dielectric coupling between the two phases of porous aggregate.

1.2. Equations of motion

Using variational principle, we can write

$$\delta\left[\int_{t_0}^{t_1} dt \int_V (K - W)\,\mathrm{d}V + \int_{t_0}^{t_1} \mathrm{d}t \int_S (t_j\delta u_j + t_j^*\delta U_j^* + c\delta\varphi + c^*\delta\varphi^*)\,\mathrm{d}S\right] = 0, \tag{7}$$

where t_j, t_j^* are surface traction for solid and fluid phase of porous bulk material. K is the kinetic energy density. c, c^* are surface charge density for solid and fluid phase.

The kinetic energy density K is defined as

$$K = \tfrac{1}{2}\{\rho_{11}\dot{u}_i\dot{u}_i + 2\rho_{12}\dot{u}_i\dot{U}_i^* + \rho_{22}\dot{U}_i^*\dot{U}_i^*\}, \tag{8}$$

where ρ_{11}, ρ_{12} and ρ_{22} are the dynamical coefficients which depend upon the porosity (f), density of porous aggregate (ρ), pore fluid density (ρ_f) and the inertial coupling parameters.

Eq. (8) implies

$$\delta\int_{t_0}^{t_1} K\,\mathrm{d}t = -\left[\rho_{11}\int_{t_0}^{t_1}\ddot{u}_i\delta u_i\,\mathrm{d}t + \rho_{22}\int_{t_0}^{t_1}\ddot{U}_i^*\delta U_i^*\,\mathrm{d}t + \rho_{12}\int_{t_0}^{t_1}(\ddot{u}_i\delta U_i^* + \ddot{U}_i^*\delta u_i)\,\mathrm{d}t\right]. \tag{9}$$

Also

$$\begin{aligned}\delta\int_V W\,\mathrm{d}V &= \int_V\left(\frac{\partial W}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{\partial W}{\partial\varepsilon^*}\delta\varepsilon^* + \frac{\partial W}{\partial E_i}\delta E_i + \frac{\partial W}{\partial E_i^*}\delta E_i^*\right)\mathrm{d}V\\ &= \int_S(\sigma_{ij}\delta u_j + \sigma^*\delta U_i^* + D_i\delta\varphi + D_i^*\delta\varphi^*)n_i\,\mathrm{d}S - \int_V(\sigma_{ij,i}\delta u_j + \sigma^*_{,i}\delta U_i^* + D_{i,i}\delta\varphi + D^*_{i,i}\delta\varphi^*)\,\mathrm{d}V.\end{aligned} \tag{10}$$

Making use of Eqs. (9) and (10), Eq. (7) can be written as

$$\begin{aligned}&\int_{t_0}^{t_1}\mathrm{d}t\int_V[(\sigma_{ij,i} - \rho_{11}\ddot{u}_j - \rho_{12}\ddot{U}_j^*)\delta u_j + (\sigma^*_{,i} - \rho_{22}\ddot{U}_i^* - \rho_{12}\ddot{u}_i)\delta U_i^* + D_{i,i}\delta\varphi + D^*_{i,i}\delta\varphi^*]\,\mathrm{d}V\\ &+\int_{t_0}^{t_1}\mathrm{d}t\int_S[(t_j - \sigma_{ij}n_i)\delta u_j + (t_i^* - \sigma^* n_i)\delta U_i^* + (c - D_i n_i)\delta\varphi + (c^* - D_i^* n_i)\delta\varphi^*]\,\mathrm{d}S = 0.\end{aligned}$$

⇒In V,

$$\sigma_{ij,i} = \rho_{11}\ddot{u}_j + \rho_{12}\ddot{U}_j^*, \tag{11a}$$

$$\sigma^*_{,i} = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i^*, \tag{11b}$$

$$D_{i,i} = 0, \tag{11c}$$

$$D^*_{i,i} = 0. \tag{11d}$$

With boundary conditions specified on the surface S as

$$\sigma_{ij}n_i = t_j, \tag{12a}$$

$$\sigma^* n_i = t_i^*, \tag{12b}$$

$$D_i n_i = c, \tag{12c}$$

$$D_i^* n_i = c^*. \tag{12d}$$

Eqs. (11a) and (11b) are equations of motion for porous piezoelectric materials when body forces are absent. Eqs. (11c) and (11d) correspond to Gauss equation.

2. Two-dimensional equations

Let us consider a plate of thickness $2b$, with Cartesian coordinate axes x_1, x_3 in the middle plane and x_2 normal to the plate (Fig. 1). Let the two-dimensional region in the $x_1 - x_3$ plane occupied by piezoelectric plate be A, the boundary curve of A be C, the unit outward normal to the curve C be n_i.

Let mechanical displacement and electrical potential can be expressed as power series in x_2:

$$\begin{aligned}&u_i = \sum_n x_2^n u_i^{(n)}, \quad U_i^* = \sum_n x_2^n U_i^{*(n)},\\ &\varphi = \sum_n x_2^n \varphi^{(n)}, \quad \varphi^* = \sum_n x_2^n \varphi^{*(n)} \quad (i = 1,2,3;\ \ n = 0,\ldots,\infty),\end{aligned} \tag{13}$$

where $u_i^{(n)}, U_i^{*(n)}, \varphi^{(n)}, \varphi^{*(n)}$ are functions of x_1, x_3 and t.

Now

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \sum_n x_2^n \varepsilon_{ij}^{(n)}. \tag{14a}$$

Similarly, we can write

$$\varepsilon^* = \sum_n x_2^{(n)}\varepsilon^{*(n)}, \tag{14b}$$

$$E_i = \sum_n x_2^{(n)} E_i^{(n)}, \tag{14c}$$

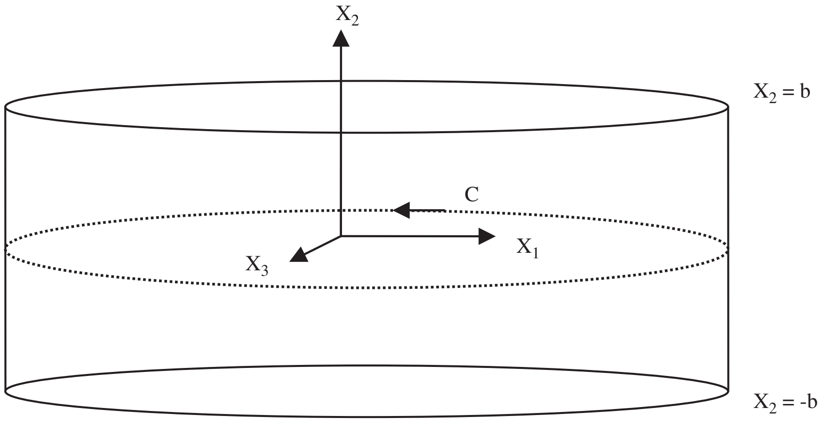

Fig. 1. Geometry of the problem.

$$E_i^* = \sum_n x_2^{(n)} E_i^{*(n)}, \tag{14d}$$

where

$$\varepsilon_{ij}^{(n)} = \tfrac{1}{2}[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)\delta_{2i}u_j^{(n+1)} + (n+1)\delta_{2j}u_i^{(n+1)}], \tag{15a}$$

$$\varepsilon^{*(n)} = U_{i,\ i}^{*(n)} + (n+1)\ \delta_{2i}\ U_i^{*(n+1)}, \tag{15b}$$

$$E_i^{(n)} = -\varphi_{,\ i}^{(n)} - (n+1)\ \delta_{2i}\ \varphi^{(n+1)}, \tag{15c}$$

$$E_i^{*(n)} = -\varphi_{,i}^{*(n)} - (n+1)\ \delta_{2i}\ \varphi^{*(n+1)}. \tag{15d}$$

Now consider

$$\begin{aligned} &\int_S (t_j\delta u_j + t_j^*\delta U_j^* + c\delta\varphi + c^*\delta\varphi^*)\,\mathrm{d}S \\ &\quad = \int_S (\sigma_{ij}\delta u_j + \sigma^*\delta_{ij}\delta U_j^* + D_i\delta\varphi + D_i^*\delta\varphi^*)n_i\,\mathrm{d}S \\ &\quad = \sum_n \int_A [x_2^n(\sigma_{2j}\delta u_j^{(n)} + \sigma^*\delta_{2j}\delta U_j^{*(n)} + D_2\delta\varphi^{(n)} + D_2^*\delta\varphi^{*(n)})]_{-b}^{+b}\,\mathrm{d}A \\ &\qquad + \sum_n \oint_C \int_{-b}^{b} x_2^n n_a(\sigma_{aj}\delta u_j^{(n)} + \sigma^*\delta_{aj}\delta U_j^{*(n)} + D_a\delta\varphi^{(n)} + D_a^*\delta\varphi^{*(n)})\,\mathrm{d}x_2\,\mathrm{d}s, \end{aligned} \tag{16}$$

where $j = 1, 2, 3$; $a = 1, 3$ and s is arc length along the curve C, on the lateral surface of the plate. A is the area of the either face $x_2 = \pm b$ of the plate.

We define

$$T_j^{(n)} = B_n^{-1}[x_2^{(n)}\sigma_{2j}]_{-b}^{b}, \quad T_j^{*(n)} = B_n^{-1}[x_2^{(n)}\sigma^*\delta_{2j}]_{-b}^{b},$$

$$D^{(n)} = B_n^{-1}[x_2^{(n)}D_2]_{-b}^{b}, \quad D^{*(n)} = B_n^{-1}[x_2^{(n)}D_2^*]_{-b}^{b},$$

$$t_j^{(n)} = B_n^{-1}\int_{-b}^{b} x_2^{(n)}(\sigma_{aj}n_a)_C\,\mathrm{d}x_2, \quad t_j^{*(n)} = B_n^{-1}\int_{-b}^{b} x_2^{(n)}(\sigma^*\delta_{aj}n_a)_C\,\mathrm{d}x_2,$$

$$d^{(n)} = B_n^{-1}\int_{-b}^{b} x_2^{(n)}(D_a n_a)_C\,\mathrm{d}x_2, \quad d^{*(n)} = B_n^{-1}\int_{-b}^{b} x_2^{(n)}(D_a^* n_a)_C\,\mathrm{d}x_2,$$

$$B_n = \frac{2b^{2n+1}}{2n+1},$$

$$\bar{K}=\int_{-b}^{b}K\,\mathrm{d}x_2,\quad \bar{W}=\int_{-b}^{b}W\,\mathrm{d}x_2. \tag{17}$$

$$\begin{aligned}\therefore \int_S (t_j\delta u_j + t_j^*\delta U_j^* + c\delta\varphi + c^*\delta\varphi^*)\,\mathrm{d}S &= \sum_n \int_A [B_n T_j^{(n)}\delta u_j^{(n)} + B_n T_j^{*(n)}\delta U_j^{*(n)} + B_n D^{(n)}\delta\varphi^{(n)} + B_n D^{*(n)}\delta\varphi^{*(n)}]\,\mathrm{d}A \\ &+ \sum_n \oint_C [B_n t_j^{(n)}\delta u_j^{(n)} + B_n t_j^{*(n)}\delta U_j^{*(n)} + B_n d^{(n)}\delta\varphi^{(n)} + B_n d^{*(n)}\delta\varphi^{*(n)}]\,\mathrm{d}s.\end{aligned}$$

Using the above equation, Eq. (7) for two-dimensional case can now be written as

$$\begin{aligned}\delta\Bigg[\int_{t_0}^{t_1}\mathrm{d}t\int_A(\bar{K}-\bar{W})\,\mathrm{d}A + \int_{t_0}^{t_1}\mathrm{d}t\int_A\sum_n B_n(T_j^{(n)}\delta u_j^{(n)} + T_j^{*(n)}\delta U_j^{*(n)} + D^{(n)}\delta\varphi^{(n)} + D^{*(n)}\delta\varphi^{*(n)})\,\mathrm{d}A \\ + \int_{t_0}^{t_1}\mathrm{d}t\oint_C\sum_n B_n(t_j^{(n)}\delta u_j^{(n)} + t_j^{*(n)}\delta U_j^{*(n)} + d^{(n)}\delta\varphi^{(n)} + d^{*(n)}\delta\varphi^{*(n)})\,\mathrm{d}s\Bigg] = 0.\end{aligned} \tag{18}$$

Here,

$$\begin{aligned}\bar{K} &= \frac{1}{2}\int_{-b}^{b}\left[\sum_m\sum_n(\rho_{11}\dot{u}_i^{(m)}\dot{u}_i^{(n)} + 2\rho_{12}\dot{u}_i^{(m)}\dot{U}_i^{*(n)} + \rho_{22}\dot{U}_i^{*(m)}\dot{U}_i^{*(n)})\right]\mathrm{d}x_2 \\ \Rightarrow \delta\int_{t_0}^{t_1}\bar{K}\,\mathrm{d}t &= -\Bigg[\rho_{11}\int_{t_0}^{t_1}\sum_m\sum_n B_{mn}\ddot{u}_i^{(m)}\delta u_i^{(n)} + \rho_{22}\int_{t_0}^{t_1}\sum_m\sum_n B_{mn}\dot{U}_i^{*(m)}\delta U_i^{*(n)} \\ &+ \rho_{12}\int_{t_0}^{t_1}\sum_m\sum_n B_{mn}(\ddot{u}_i^{(m)}\delta U_i^{*(n)} + \ddot{U}_i^{*(n)}\delta u_i^{(m)})\Bigg],\end{aligned} \tag{19}$$

where

$$B_{mn}=\int_{-b}^{b}x_2^{m+n}\,\mathrm{d}x_2=\begin{cases}\dfrac{2b^{m+n+1}}{m+n+1} & \text{when } m+n \text{ is even}\\ 0 & \text{when } m+n \text{ is odd}\end{cases},$$

and

$$\begin{aligned}\delta\bar{W} &= \int_{-b}^{b}\delta W\,\mathrm{d}x_2 = \sum_n(\sigma_{ij}^{(n)}\delta\varepsilon_{ij}^{(n)} + \sigma^{*(n)}\delta\varepsilon^{*(n)} - D_i^{(n)}\delta E_i^{(n)} - D_i^{*(n)}\delta E_i^{*(n)})\,\mathrm{d}x_2 \\ &= \sum_n[\sigma_{ij}^{(n)}\delta u_j^{(n)} + \sigma^{*(n)}\delta_{ij}\delta U_j^{*(n)} + D_i^{(n)}\delta\varphi^{(n)} + D_i^{*(n)}\delta\varphi^{*(n)}]_{,i} \\ &+ \sum_n[(n\sigma_{2j}^{(n-1)} - \sigma_{ij,i}^{(n)})\delta u_j^{(n)} + (n\sigma^{*(n-1)}\delta_{2j} - \sigma_{,j}^{*(n)})\delta U_j^{*(n)} \\ &+ (nD_2^{(n-1)} - D_{i,i}^{(n)})\delta\varphi^{(n)} + (nD_2^{*(n-1)} - D_{i,i}^{*(n)})\delta\varphi^{*(n)}],\end{aligned}$$

where

$$\sigma_{ij}^{(n)}=\int_{-b}^{b}\sigma_{ij}x_2^{(n)}\,\mathrm{d}x_2,\quad \sigma^{*(n)}=\int_{-b}^{b}\sigma^{*}x_2^{(n)}\,\mathrm{d}x_2,\quad D_i^{(n)}=\int_{-b}^{b}D_ix_2^{(n)}\,\mathrm{d}x_2,\quad D_i^{*(n)}=\int_{-b}^{b}D_i^{*}x_2^{(n)}\,\mathrm{d}x_2. \tag{20}$$

$$\begin{aligned}\therefore\int_A\delta\bar{W}\,\mathrm{d}A &= \int_A\sum_n[(n\sigma_{2j}^{(n-1)} - \sigma_{ij,i}^{(n)})\delta u_j^{(n)} + (n\sigma^{*(n-1)}\delta_{2j} - \sigma_{,j}^{*(n)})\delta U_j^{*(n)} \\ &+ (nD_2^{(n-1)} - D_{i,i}^{(n)})\delta\varphi^{(n)} + (nD_2^{*(n-1)} - D_{i,i}^{*(n)})\delta\varphi^{*(n)}]\,\mathrm{d}A \\ &+ \oint_C\sum_n[\sigma_{aj}^{(n)}\delta u_j^{(n)} + \sigma^{(n)}\delta_{aj}\delta U_j^{*(n)} + D_a^{(n)}\delta\varphi^{(n)} + D_a^{*(n)}\delta\varphi^{*(n)}]n_a\,\mathrm{d}s.\end{aligned} \tag{21}$$

Using Eqs. (19) and (21) in Eq. (18), we get

$$\sum_n \int_{t_0}^{t_1} \mathrm{d}t \int_A \left[\sigma_{ij,i}^{(n)} - n\sigma_{2j}^{(n-1)} - \sum_m \rho_{11} B_{mn} \ddot{u}_j^{(m)} - \sum_m \rho_{12} B_{mn} \ddot{U}_j^{*(m)} + B_n T_j^{(n)}\right] \delta u_j^{(n)}\, \mathrm{d}A$$
$$+ \sum_n \int_{t_0}^{t_1} \mathrm{d}t \int_A \left[\sigma_{,j}^{*(n)} - n\delta_{2j}\sigma^{*(n-1)} - \sum_m \rho_{12} B_{mn} \ddot{u}_j^{(m)} - \sum_m \rho_{22} B_{mn} \ddot{U}_j^{*(m)} + B_n T_j^{*(n)}\right] \delta U_j^{*(n)}\, \mathrm{d}A$$
$$+ \sum_n \int_{t_0}^{t_1} \mathrm{d}t \int_A (D_{i,i}^{(n)} - nD_2^{(n-1)} + B_n D^{(n)}) \delta\varphi^{(n)}\, \mathrm{d}A$$
$$+ \sum_n \int_{t_0}^{t_1} \mathrm{d}t \int_A (D_{i,i}^{*(n)} - nD_2^{*(n-1)} + B_n D^{*(n)}) \delta\varphi^{*(n)}\, \mathrm{d}A$$
$$+ \sum_n \int_{t_0}^{t_1} \mathrm{d}t \oint_C [(B_n t_j^{(n)} - \sigma_{aj}^{(n)} n_a)\delta u_j^{(n)} + (B_n T_j^{*(n)} - \sigma^{*(n)}\delta_{aj} n_a)\delta U_j^{*(n)}$$
$$+ (B_n d^{(n)} - D_a^{(n)} n_a)\delta\varphi^{(n)} + (B_n d^{*(n)} - D_a^{*(n)} n_a)\delta\varphi^{*(n)}]\, \mathrm{d}s = 0.$$

$$\Rightarrow \text{In A}, \quad \sigma_{ij,i}^{(n)} - n\sigma_{2j}^{(n-1)} + B_n T_j^{(n)} = \sum_m B_{mn}(\rho_{11}\ddot{u}_j^{(m)} + \rho_{12}\ddot{U}_j^{*(m)}), \tag{22a}$$

$$\sigma_{,j}^{*(n)} - n\sigma*^{(n-1)}\delta_{2j} + B_n T_j^{*(n)} = \sum_m B_{mn}(\rho_{12}\ddot{u}_j^{(m)} + \rho_{22}\ddot{U}_j^{*(m)}), \tag{22b}$$

$$D_{i,i}^{(n)} - nD_2^{(n-1)} + B_n D^{(n)} = 0, \tag{22c}$$

$$D_{i,i}^{(n)} - nD_2^{*(n-1)} + B_n D^{*(n)} = 0. \tag{22d}$$

With boundary conditions on the curve C,

$$B_n t_j^{(n)} - \sigma_{aj}^{(n)} n_a = 0, \tag{23a}$$

$$B_n t_j^{*(n)} - \sigma^{*(n)}\delta_{aj} n_a = 0, \tag{23b}$$

$$B_n d^{(n)} - D_a^{(n)} n_a = 0, \tag{23c}$$

$$B_n d^{*(n)} - D_a^{*(n)} n_a = 0 \tag{23d}$$

The constitutive equations of order n are

$$\sigma_{ij}^{(n)} = \int_{-b}^{b} x_2^{(n)} \sigma_{ij}\, \mathrm{d}x_2 = \int_{-b}^{b} \sum_m x_2^n x_2^m (c_{ijkl}\varepsilon_{kl}^{(m)} + m_{ij}\varepsilon^{*(m)} - e_{kij}E_k^{(m)} - \zeta_{kij}E_k^{*(m)})\, \mathrm{d}x_2.$$

Since $\varepsilon_{kl}^{(m)}$, $\varepsilon^{*(m)}$, $E_k^{(m)}$, $E_k^{*(m)}$ are independent of x_2, therefore above integral can be written as

$$\sigma_{ij}^{(n)} = \sum_m B_{mn}(c_{ijkl}\varepsilon_{kl}^{(m)} + m_{ij}\varepsilon^{*(m)} - e_{kij}E_k^{(m)} - \zeta_{kij}E_k^{*(m)}). \tag{24a}$$

Similarly, other constitutive equations can be written as

$$\sigma^{*(n)} = \sum_m B_{mn}(m_{ij}\varepsilon_{ij}^{(m)} + R\varepsilon^{*(m)} - \tilde{\zeta}_i E_i^{(m)} - e_i^* E_i^{*(m)}), \tag{24b}$$

$$D_i^{(n)} = \sum_m B_{mn}(e_{ijk}\varepsilon_{jk}^{(m)} + \tilde{\zeta}_i\varepsilon^{*(m)} + \xi_{ij}E_j^{(m)} + A_{ij}E_j^{*(m)}), \tag{24c}$$

$$D_i^{*(n)} = \sum_m B_{mn}(\zeta_{ijk}\varepsilon_{jk}^{(m)} + e_i^*\varepsilon^{*(m)} + A_{ij}E_j^{(m)} + \xi_{ij}^* E_j^{*(m)}). \tag{24d}$$

3. Truncation of series and adjustments

After truncation of terms of order higher than 1, Eqs. (24a)–(24d) for the cases $n = 0$ and 1 can now be written as

$$\sigma_{ij}^{(0)} = 2b(c_{ijkl}\varepsilon_{kl}^{(0)} + m_{ij}\varepsilon^{*(0)} - e_{kij}E_k^{(0)} - \zeta_{kij}E_k^{*(0)}), \tag{25a}$$

$$\sigma^{*(0)} = 2b(m_{ij}\varepsilon_{ij}^{(0)} + R\varepsilon^{*(0)} - \tilde{\zeta}_i E_i^{(0)} - e_i^* E_i^{*(0)}), \tag{25b}$$

$$D_i^{(0)} = 2b(e_{ijk}\varepsilon_{jk}^{(0)} + \tilde{\zeta}_i\varepsilon^{*(0)} + \xi_{ij}E_j^{(0)} + A_{ij}E_j^{*(0)}), \tag{25c}$$

$$D_i^{*(0)} = 2b(\zeta_{ijk}\varepsilon_{jk}^{(0)} + e_i^*\varepsilon^{*(0)} + A_{ij}E_j^{(0)} + \xi_{ij}^* E_j^{*(0)}) \tag{25d}$$

and

$$\sigma_{ij}^{(1)} = \frac{2b^3}{3}(c_{ijkl}\varepsilon_{kl}^{(1)} + m_{ij}\varepsilon^{*(1)} - e_{kij}E_k^{(1)} - \zeta_{kij}E_k^{*(1)}), \tag{26a}$$

$$\sigma^{*(1)} = \frac{2b^3}{3}(m_{ij}\varepsilon_{ij}^{(1)} + R\varepsilon^{*(1)} - \tilde{\zeta}_i E_i^{(1)} - e_i^* E_i^{*(1)}), \tag{26b}$$

$$D_i^{(1)} = \frac{2b^3}{3}(e_{ijk}\varepsilon_{jk}^{(1)} + \tilde{\zeta}_i\varepsilon^{*(1)} + \xi_{ij}E_j^{(1)} + A_{ij}E_j^{*(1)}), \tag{26c}$$

$$D_i^{*(1)} = \frac{2b^3}{3}(\zeta_{ijk}\varepsilon_{jk}^{(1)} + e_i^*\varepsilon^{*(1)} + A_{ij}E_j^{(1)} + \xi_{ij}^* E_j^{*(1)}) \tag{26d}$$

The kinetic energy density $\bar{K}$ after truncation of terms of order higher than 1 is

$$\bar{K} = b(\rho_{11}\dot{u}_i^{(0)}\dot{u}_i^{(0)} + 2\rho_{12}\dot{u}_i^{(0)}\dot{U}_i^{*(0)} + \rho_{22}\dot{U}_i^{*(0)}\dot{U}_i^{*(0)}) + \frac{b^3}{3}(\rho_{11}\dot{u}_i^{(1)}\dot{u}_i^{(1)} + 2\rho_{12}\dot{u}_i^{(1)}\dot{U}_i^{*(1)} + \rho_{22}\dot{U}_i^{*(1)}\dot{U}_i^{*(1)}). \tag{27}$$

Following Cauchy [46], we neglect the velocity $\dot{u}_2^{(1)}$, $\dot{U}_2^{(1)}$ in kinetic energy density and free development of strain $\varepsilon_{22}^{(0)}$ is obtained by setting $\sigma_{22}^{(0)} = 0$ in (25a) i.e.

$$\varepsilon_{22}^{(0)} = \frac{-c_{22kl}}{c_{2222}}\varepsilon_{kl}^{(0)} - \frac{m_{22}}{c_{2222}}\varepsilon^{*(0)} + \frac{e_{k22}}{c_{2222}}E_k^{(0)} + \frac{\zeta_{k22}}{c_{2222}}E_k^{*(0)} + \varepsilon_{22}^{(0)}. \tag{28}$$

$$\Rightarrow (2b)^{-1}\sigma_{ij}^{(0)} = (c_{ijkl}\varepsilon_{kl}^{(0)} - c_{ij22}\varepsilon_{22}^{(0)}) + c_{ij22}\varepsilon_{22}^{(0)} + m_{ij}\varepsilon^{*(0)} - e_{kij}E_k^{(0)} - \zeta_{kij}E_k^{*(0)}.$$

Using Eq. (28), above equation reduces to

$$\sigma_{ij}^{(0)} = 2b(\bar{c}_{ijkl}\varepsilon_{kl}^{(0)} + \bar{m}_{ij}\varepsilon^{*(0)} - \bar{e}_{kij}E_k^{(0)} - \bar{\zeta}_{kij}E_k^{*(0)}), \tag{29a}$$

where

$$\bar{c}_{ijkl} = c_{ijkl} - \frac{c_{ij22}c_{22kl}}{c_{2222}}, \quad \bar{m}_{ij} = m_{ij} - \frac{c_{ij22}m_{22}}{c_{2222}}, \quad \bar{e}_{kij} = e_{kij} - \frac{c_{ij22}e_{k22}}{c_{2222}}, \quad \bar{\zeta}_{kij} = \zeta_{kij} - \frac{c_{ij22}\zeta_{k22}}{c_{2222}}.$$

By following same procedure of elimination, the expressions for $\sigma^{*(0)}, D_i^{(0)}, D_i^{*(0)}$ can be obtained as

$$\sigma^{*(0)} = 2b(\bar{m}_{ij}\varepsilon_{ij}^{(0)} + \bar{R}\varepsilon^{*(0)} - \bar{\tilde{\zeta}}_i E_i^{(0)} - \bar{e}_i^* E_i^{*(0)}), \tag{29b}$$

$$D_i^{(0)} = 2b(\bar{e}_{ijk}\varepsilon_{jk}^{(0)} + \bar{\tilde{\zeta}}_i\varepsilon^{*(0)} + \bar{\xi}_{ij}E_j^{(0)} + \bar{A}_{ij}E_j^{*(0)}), \tag{29c}$$

$$D_i^{*(0)} = 2b(\bar{\zeta}_{ijk}\varepsilon_{jk}^{(0)} + \bar{e}_i^*\varepsilon^{*(0)} + \bar{A}_{ij}E_j^{(0)} + \bar{\xi}_{ij}^* E_j^{*(0)}), \tag{29d}$$

where

$$\bar{\xi}_{ij} = \xi_{ij} + \frac{e_{i22}e_{j22}}{c_{2222}}, \quad \bar{\xi}_{ij}^* = \xi_{ij}^* + \frac{\zeta_{i22}\zeta_{j22}}{c_{2222}},$$

$$\bar{A}_{ij} = A_{ij} + \frac{e_{i22}\zeta_{j22}}{c_{2222}}, \quad \bar{R} = R - \frac{m_{22}m_{22}}{c_{2222}},$$

$$\bar{e}_i^* = e_i^* - \frac{m_{22}\zeta_{i22}}{c_{2222}}, \quad \bar{\tilde{\zeta}}_i = \tilde{\zeta}_i - \frac{m_{22}e_{i22}}{c_{2222}}.$$

In the first-order terms, we neglect the velocity $\dot{u}_j^{(2)}$, $\dot{U}_j^{(2)}$ in kinetic energy density and free development of strains $\varepsilon_{2j}^{(1)}$ is obtained by setting $\sigma_{2j}^{(1)}=0$ in Eq. (26a) which gives

$$s_{ijmn}\sigma_{ij}^{(1)}=\frac{2b^3}{3}(s_{ijmn}c_{ijkl}\varepsilon_{kl}^{(1)}+s_{ijmn}m_{ij}\varepsilon^{*(1)}-s_{ijmn}e_{kij}E_k^{(1)}-s_{ijmn}\zeta_{kij}E_k^{*(1)}), \tag{30}$$

where s_{ijmn} is the compliance tensor. We define

$$I_{mnkl}=c_{ijkl}s_{ijmn},\quad m_{mn}^I=s_{ijmn}m_{ij},\quad e_{kmn}^I=s_{ijmn}e_{kij},\quad \zeta_{kmn}^I=s_{ijmn}\zeta_{kij},$$

$$s_{ijmn}\sigma_{ij}^{(1)}=\frac{2b^3}{3}(\varepsilon_{mn}^{(1)}+m_{mn}^I\varepsilon^{*(1)}-e_{kmn}^IE_k^{(1)}-\zeta_{kmn}^IE_k^{*(1)}).$$

By setting $\sigma_{2j}^{(1)}=0$ in the above equation, we get

$$s_{abcd}\sigma_{cd}^{(1)}=\frac{2b^3}{3}(\varepsilon_{ab}^{(1)}+m_{ab}^I\varepsilon^{*(1)}-e_{cab}^IE_c^{(1)}-\zeta_{cab}^IE_c^{*(1)})\quad (a,b,c,d=1 \text{ and } 3).$$

Solving this system of equations for$\sigma_{11}^{(1)}$, $\sigma_{13}^{(1)}$, $\sigma_{33}^{(1)}$, we obtain

$$\sigma_{ab}^{(1)}=\frac{2b^3A_{abcd}}{3|s_{abcd}|}(\varepsilon_{cd}^{(1)}+m_{cd}^I\varepsilon^{*(1)}-e_{ecd}^IE_e^{(1)}-\zeta_{ecd}^IE_e^{*(1)}),$$

where

$$|s_{abcd}|=\begin{vmatrix} s_{1111} & s_{3311} & s_{1311}\\ s_{1133} & s_{3333} & s_{1333}\\ s_{1113} & s_{3313} & s_{1313}\end{vmatrix},$$

and A_{abcd} is corresponding cofactor of element s_{abcd} in $[s_{abcd}]$.

$$\therefore \sigma_{ab}^{(1)}=\frac{2b^3}{3}(c_{abcd}^{(1)}\varepsilon_{cd}^{(1)}+m_{ab}^{(1)}\varepsilon^{*(1)}-e_{cab}^{(1)}E_c^{(1)}-\zeta_{cab}^{(1)}E_c^{*(1)}), \tag{31}$$

where

$$c_{abcd}^{(1)}=\frac{A_{abcd}}{|s_{abcd}|},\quad m_{ab}^{(1)}=c_{abcd}^{(1)}m_{cd}^I,\quad e_{eab}^{(1)}=c_{abcd}^1e_{ecd}^I,\quad \zeta_{eab}^{(1)}=c_{abcd}^1\zeta_{ecd}^I.$$

The electric enthalpy density function $\bar{W}$ from which relations (29a) and (31) are obtained, can now be written as

$$\begin{aligned}\bar{W}=\,&b[\bar{c}_{ijkl}\varepsilon_{ij}^{(0)}\varepsilon_{kl}^{(0)}+2\bar{m}_{ij}\varepsilon_{ij}^{(0)}\varepsilon^{*(0)}+\bar{R}\varepsilon^{*(0)}\varepsilon^{*(0)}-2\bar{e}_{kij}\varepsilon_{ij}^{(0)}E_k^{(0)}-2\bar{\zeta}_{kij}\varepsilon_{ij}^{(0)}E_k^{*(0)}\\&-2\bar{\bar{\zeta}}_iE_i^{(0)}\varepsilon^{*(0)}-2\bar{e}_i^*E_i^{*(0)}\varepsilon^{*(0)}-\bar{\xi}_{ij}E_i^{(0)}E_j^{(0)}-\bar{\xi}_{ij}^*E_i^{*(0)}E_j^{*(0)}-2\bar{A}_{ij}E_i^{(0)}E_j^{*(0)}]\\&+\frac{b^3}{3}[c_{abcd}^{(1)}\varepsilon_{ab}^{(1)}\varepsilon_{cd}^{(1)}+2m_{ab}^{(1)}\varepsilon_{ab}^{(1)}\varepsilon^{*(1)}+R\varepsilon^{*(1)}\varepsilon^{*(1)}-2e_{cab}^{(1)}\varepsilon_{ab}^{(1)}E_c^{(1)}-2\zeta_{cab}^{(1)}\varepsilon_{ab}^{(1)}E_c^{*(1)}\\&-2\tilde{\zeta}_aE_a^{(1)}\varepsilon^{*(1)}-2e_a^*E_a^{*(1)}\varepsilon^{*(1)}-\xi_{ab}E_a^{(1)}E_b^{(1)}-\xi_{ab}^*E_a^{*(1)}E_b^{*(1)}-2A_{ab}E_a^{(1)}E_b^{*(1)}].\end{aligned} \tag{32}$$

The final adjustment is made by replacing thickness shear strains $\varepsilon_{21}^{(0)}$, $\varepsilon_{23}^{(0)}$ by $k_1\varepsilon_{21}^{(0)}$, $k_3\varepsilon_{23}^{(0)}$, respectively, where k_1, k_3 are thickness shear correction factors whose values are to be determined. This gives

$$\begin{aligned}\bar{W}=\,&b[c_{ijkl}^{(0)}\varepsilon_{ij}^{(0)}\varepsilon_{kl}^{(0)}+2m_{ij}^{(0)}\varepsilon_{ij}^{(0)}\varepsilon^{*(0)}+\bar{R}\varepsilon^{*(0)}\varepsilon^{*(0)}-2e_{kij}^{(0)}\varepsilon_{ij}^{(0)}E_k^{(0)}-2\zeta_{kij}^{(0)}\varepsilon_{ij}^{(0)}E_k^{*(0)}\\&-2\bar{\bar{\zeta}}_iE_i^{(0)}\varepsilon^{*(0)}-2\bar{e}_i^*E_i^{*(0)}\varepsilon^{*(0)}-\bar{\xi}_{ij}E_i^{(0)}E_j^{(0)}-\bar{\xi}_{ij}^*E_i^{*(0)}E_j^{*(0)}-2\bar{A}_{ij}E_i^{(0)}E_j^{*(0)}]\\&+\frac{b^3}{3}[c_{abcd}^{(1)}\varepsilon_{ab}^{(1)}\varepsilon_{cd}^{(1)}+2m_{ab}^{(1)}\varepsilon_{ab}^{(1)}\varepsilon^{*(1)}+R\varepsilon^{*(1)}\varepsilon^{*(1)}-2e_{cab}^{(1)}\varepsilon_{ab}^{(1)}E_c^{(1)}-2\zeta_{cab}^{(1)}\varepsilon_{ab}^{(1)}E_c^{*(1)}\\&-2\tilde{\zeta}_aE_a^{(1)}\varepsilon^{*(1)}-2e_a^*E_a^{*(1)}\varepsilon^{*(1)}-\xi_{ab}E_a^{(1)}E_b^{(1)}-\xi_{ab}^*E_a^{*(1)}E_b^{*(1)}-2A_{ab}E_a^{(1)}E_b^{*(1)}],\end{aligned} \tag{33}$$

where

$$c_{ijkl}^{(0)}=k'_{i+j-2}k'_{k+l-2}\bar{c}_{ijkl},\quad m_{ij}^{(0)}=k'_{i+j-2}\bar{m}_{ij},\quad e_{kij}^{(0)}=k'_{i+j-2}\bar{e}_{kij},\quad \zeta_{kij}^{(0)}=k'_{i+j-2}\bar{\zeta}_{kij},$$

$$k'_{i+j-2}\ or\ (k'_{k+l-2}) = \begin{cases} k_1 & \text{when } (i+j) \text{ or } (k+l) = 3 \\ k_3 & \text{when } (i+j) \text{ or } (k+l) = 5 \\ 1 & \text{otherwise.} \end{cases}$$

After truncation, Eqs. (14a)–(14d), (15a)–(15d), (29a)–(29d), Eq. (31) can be rewritten as

Strain–displacement relation:

$$\varepsilon_{ij}^{(0)} = \tfrac{1}{2}[u_{i,j}^{(0)} + u_{j,i}^{(0)} + \delta_{2i}u_j^{(1)} + \delta_{2j}u_i^{(1)}],$$

$$\varepsilon_{ab}^{(1)} = \tfrac{1}{2}[u_{a,b}^{(1)} + u_{b,a}^{(1)}],$$

$$\varepsilon^{*(0)} = U_{i,i}^{*(0)} + \delta_{2i}U_i^{*(1)},$$

$$\varepsilon^{*(1)} = U_{a,a}^{*(1)},$$

$$E_i^{(0)} = -\varphi_{,i}^{(0)} - \delta_{2i}\varphi^{(1)},$$

$$E_i^{*(0)} = -\varphi_{,i}^{*(0)} - \delta_{2i}\varphi^{*(1)},$$

$$E_a^{(1)} = -\varphi_{,a}^{(1)},$$

$$E_a^{*(1)} = -\varphi_{,a}^{*(1)}. \tag{34}$$

Kinetic energy density:

$$\bar{K} = b(\rho_{11}\dot{u}_i^{(0)}\dot{u}_i^{(0)} + 2\rho_{12}\dot{u}_i^{(0)}\dot{U}_i^{*(0)} + \rho_{22}\dot{U}_i^{*(0)}\dot{U}_i^{*(0)}) + \frac{b^3}{3}(\rho_{11}\dot{u}_a^{(1)}\dot{u}_a^{(1)} + 2\rho_{12}\dot{u}_a^{(1)}\dot{U}_a^{*(1)} + \rho_{22}\dot{U}_a^{*(1)}\dot{U}_a^{*(1)}). \tag{35}$$

Constitutive equations:

$$\sigma_{ij}^{(0)} = 2b(c_{ijkl}^{(0)}\varepsilon_{kl}^{(0)} + m_{ij}^{(0)}\varepsilon^{*(0)} - e_{kij}^{(0)}E_k^{(0)} - \zeta_{kij}^{(0)}E_k^{*(0)}), \tag{36a}$$

$$\sigma^{*(0)} = 2b(m_{ij}^{(0)}\varepsilon_{ij}^{(0)} + \bar{R}\varepsilon^{*(0)} - \bar{\bar{\zeta}}_i E_i^{(0)} - \bar{e}_i^* E_i^{*(0)}), \tag{36b}$$

$$D_i^{(0)} = 2b(e_{ijk}^{(0)}\varepsilon_{jk}^{(0)} + \bar{\bar{\zeta}}_i\varepsilon^{*(0)} + \bar{\xi}_{ij}E_j^{(0)} + \bar{A}_{ij}E_j^{*(0)}), \tag{36c}$$

$$D_i^{*(0)} = 2b(\zeta_{ijk}^{(0)}\varepsilon_{jk}^{(0)} + \bar{e}_i^*\varepsilon^{*(0)} + \bar{A}_{ij}E_j^{(0)} + \bar{\xi}_{ij}^* E_j^{*(0)}) \tag{36d}$$

and

$$\sigma_{ab}^{(1)} = \frac{2b^3}{3}(c_{abcd}^{(1)}\varepsilon_{cd}^{(1)} + m_{ab}^{(1)}\varepsilon^{*(1)} - e_{cab}^{(1)}E_c^{(1)} - \zeta_{cab}^{(1)}E_c^{*(1)}), \tag{37a}$$

$$\sigma^{*(1)} = \frac{2b^3}{3}(m_{ab}^{(1)}\varepsilon_{ab}^{(1)} + R\varepsilon^{*(1)} - \tilde{\zeta}_a E_a^{(1)} - e_a^* E_a^{*(1)}), \tag{37b}$$

$$D_a^{(1)} = \frac{2b^3}{3}(e_{abc}^{(1)}\varepsilon_{bc}^{(1)} + \tilde{\zeta}_a\varepsilon^{*(1)} + \xi_{ab}E_b^{(1)} + A_{ab}E_b^{*(1)}), \tag{37c}$$

$$D_a^{*(1)} = \frac{2b^3}{3}(\zeta_{abc}^{(1)}\varepsilon_{bc}^{(1)} + e_a^*\varepsilon^{*(1)} + A_{ab}E_b^{(1)} + \xi_{ab}^* E_b^{*(1)}). \tag{37d}$$

Equations of motion:

$$\sigma_{ij,i}^{(0)} + 2bT_j^{(0)} = 2b(\rho_{11}\ddot{u}_j^{(0)} + \rho_{12}\ddot{U}_j^{*(0)}), \tag{38a}$$

$$\sigma_{,j}^{*(0)} + 2bT_j^{*(0)} = 2b(\rho_{12}\ddot{u}_j^{(0)} + \rho_{22}\ddot{U}_j^{*(0)}), \tag{38b}$$

$$D_{i,i}^{(0)} + 2bD^{(0)} = 0, \tag{38c}$$

$$D_{i,i}^{*(0)} + 2bD^{*(0)} = 0 \tag{38d}$$

and

$$\sigma_{ab,a}^{(1)} - \sigma_{2b}^{(0)} + \frac{2b^3}{3}T_b^{(1)} = \frac{2b^3}{3}(\rho_{11}\ddot{u}_b^{(1)} + \rho_{12}\ddot{U}_b^{*(1)}), \tag{39a}$$

$$\sigma_{,b}^{*(1)} - \sigma^{*(0)}\delta_{2b} + \frac{2b^3}{3}T_b^{*(1)} = \frac{2b^3}{3}(\rho_{12}\ddot{u}_b^{(1)} + \rho_{22}\ddot{U}_b^{*(1)}), \tag{39b}$$

$$D_{a,a}^{(1)} + \frac{2b^3}{3}D^{(1)} - D_2^{(0)} = 0, \tag{39c}$$

$$D_{a,a}^{*(1)} + \frac{2b^3}{3}D^{*(1)} - D_2^{*(0)} = 0. \tag{39d}$$

4. Thickness shear coefficients

The thickness shear coefficients k_1 and k_3 are determined by equating thickness shear frequencies for the two-dimensional case to the corresponding frequencies for the three-dimensional case. Using constitutive Eqs. (36) and (37) for monoclinic (2) porous piezoelectric material (considering x_2–x_3 as mirror plane), Eqs. (38a)–(38d) and (39a)–(39d), after simplification and setting all spatial derivatives to zero, becomes

$$-\frac{3}{b^2}[c_{66}^{(0)}u_1^{(1)} + e_{26}^{(0)}\phi^{(1)} + \zeta_{26}^{(0)}\phi^{*(1)}] + T_1^{(1)} = \rho_{11}\ddot{u}_1^{(1)} + \rho_{12}\ddot{U}_1^{*(1)}, \tag{40a}$$

$$-\frac{3}{b^2}[c_{44}^{(0)}u_3^{(1)}] + T_3^{(1)} = \rho_{11}\ddot{u}_3^{(1)} + \rho_{12}\ddot{U}_3^{*(1)}, \tag{40b}$$

$$T_1^{*(1)} = \rho_{12}\ddot{u}_1^{(1)} + \rho_{22}\ddot{U}_1^{*(1)}, \tag{40c}$$

$$T_3^{*(1)} = \rho_{12}\ddot{u}_3^{(1)} + \rho_{22}\ddot{U}_3^{*(1)}, \tag{40d}$$

$$-\frac{3}{b^2}[e_{26}^{(0)}u_1^{(1)} - \bar{\xi}_{22}\phi^{(1)} - \bar{A}_{22}\phi^{*(1)}] + D^{(1)} = 0, \tag{40e}$$

$$-\frac{3}{b^2}[\zeta_{26}^{(0)}u_1^{(1)} - \bar{A}_{22}\phi^{(1)} - \bar{\xi}_{22}^{*}\phi^{*(1)}] + D^{*(1)} = 0. \tag{40f}$$

The notations used in the above equations are given in Appendix A.

Elimination of $U_1^{*(1)}$ and $U_3^{*(1)}$, gives

$$-\frac{3}{b^2}[c_{66}^{(0)}u_1^{(1)} + e_{26}^{(0)}\phi^{(1)} + \zeta_{26}^{(0)}\phi^{*(1)}] + T_1^{(1)} = \bar{\rho}\ddot{u}_1^{(1)}, \tag{41a}$$

$$-\frac{3}{b^2}c_{44}^{(0)}u_3^{(1)} + T_3^{(1)} = \bar{\rho}\ddot{u}_3^{(1)}, \tag{41b}$$

$$-\frac{3}{b^2}[e_{26}^{(0)}u_1^{(1)} - \bar{\xi}_{22}\phi^{(1)} - \bar{A}_{22}\phi^{*(1)}] + D^{(1)} = 0, \tag{41c}$$

$$-\frac{3}{b^2}[\zeta_{26}^{(0)}u_1^{(1)} - \bar{A}_{22}\phi^{(1)} - \bar{\xi}_{22}^{*}\phi^{*(1)}] + D^{*(1)} = 0, \tag{41d}$$

where

$$\bar{\rho} = \rho_{11} - \frac{\rho_{12}^2}{\rho_{22}}.$$

Let alternating voltage $Ve^{i\omega t}$ be applied to electrode film deposit on each face of plate. For the case of thickness-shear vibrations in x_1 direction, the surface conditions are

$$\sigma_{21}|_{\pm b} = \mp 2\rho' b' \ddot{u}_1|_{\pm b}, \quad \phi = Ve^{i\omega t}, \quad \phi^* = Ve^{i\omega t}, \tag{42}$$

where $2\rho' b'$ is mass per unit area of each electrode film. Making use of Eqs. (14) and (17), these boundary conditions give

$$T_1^{(1)} = -\frac{3}{b}R'\bar{\rho}\ddot{u}_1, \quad \phi^{(1)} = \frac{V}{b}e^{i\omega t}, \quad \phi^{*(1)} = \frac{V}{b}e^{i\omega t}, \quad \text{where } R' = \frac{2\rho' b'}{\bar{\rho} b}. \tag{43}$$

Making use of $u_1^{(1)} = Ae^{i\omega t}$ and Eq. (43) in Eq. (41a), we obtain

$$A = \frac{3k_1(e_{26}+\zeta_{26})V}{b[(3R'+1)b^2\bar{\rho}\omega^2 - 3k_1^2 c_{66}]}. \tag{44}$$

Resonance occur when

$$\omega^2 = 3k_1^2 c_{66}/\bar{\rho} b^2(3R'+1). \tag{45}$$

In the absence of electrode film

$$T_1^{(1)} = 0, \quad D_1^{(1)} = 0, \quad D_1^{*(1)} = 0. \tag{46}$$

Using these conditions, Eqs. (41a), (41c) and (41d) reduce to

$$-\frac{3}{b^2}[c_{66}^{(0)}u_1^{(1)} + e_{26}^{(0)}\phi^{(1)} + \zeta_{26}^{(0)}\phi^{*(1)}] = \bar{\rho}\ddot{u}_1^{(1)},$$

$$-\frac{3}{b^2}[e_{26}^{(0)}u_1^{(1)} - \bar{\xi}_{22}\phi^{(1)} - \bar{A}_{22}\phi^{*(1)}] = 0,$$

$$-\frac{3}{b^2}[\zeta_{26}^{(0)}u_1^{(1)} - \bar{A}_{22}\phi^{(1)} - \bar{\xi}_{22}^{*}\phi^{*(1)}] = 0.$$

Substituting $u_1^{(1)} = A'e^{i\omega t}$, $\phi^{(1)} = B'e^{i\omega t}$, $\phi^{*(1)} = C'e^{i\omega t}$, in the above equation and eliminating A', B', C', we get

$$\omega^2 = \frac{3k_1^2\hat{c}_{66}}{\bar{\rho} b^2} \quad \text{(without electrode)}, \tag{47}$$

where

$$\hat{c}_{66} = c_{66} + \frac{e_{26}^2\xi_{22}^{*} + \zeta_{26}^2\xi_{22} - 2A_{22}e_{26}\zeta_{26}}{\xi_{22}\xi_{22}^{*} - A_{22}^2}.$$

For the case of thickness shear in x_3 direction, the surface conditions become

$$T_3^{(1)} = -\frac{3}{b}R'\bar{\rho}\ddot{u}_3, \quad u_3^{(1)} = A''e^{i\omega t} \tag{48}$$

Use of above conditions in Eq. (41b) implies that

$$\omega^2 = \frac{3k_3^2 c'_{44}}{b^2\bar{\rho}(3R'+1)} \quad \text{(with electrode)} \tag{49}$$

and

$$\omega^2 = \frac{3k_3^2 c'_{44}}{b^2\bar{\rho}} \quad \text{(without electrode)}, \tag{50}$$

where

$$c'_{44} = c_{44} - \frac{c_{42}c_{24}}{c_{22}}.$$

Next we will find thickness-shear frequency for the three-dimensional case. For the case of thickness shear vibration in x_1 direction, the appropriate equations of motion and boundary conditions are

$$c_{66}u_{1,22} + e_{26}\phi_{,22} + \zeta_{26}\phi^{*}_{,22} = \rho_{11}\ddot{u}_1 + \rho_{12}\ddot{U}_1^{*},$$

$$\rho_{12}\ddot{u}_1 + \rho_{22}\ddot{U}_1^{*} = 0,$$

$$e_{26}u_{1,12} - \xi_{22}\phi_{,22} - A_{22}\phi^{*}_{,22} = 0,$$

$$\zeta_{26}u_{1,22} - A_{22}\phi_{,22} - \xi_{22}^{*}\phi^{*}_{,22} = 0; \tag{51}$$

$$c_{66}u_{1,2} + e_{26}\phi_{,2} + \zeta_{26}\phi^{*}_{,2} \mp 2\rho' b'\omega^2 u_1 = 0, \quad \text{at } x_2 = \pm b, \tag{52a}$$

$$\phi, \phi^* = \pm\phi^0, \quad \text{at } x_2 = \pm b. \tag{52b}$$

The system (51) implies that

$$c_{66}u_{1,22} + e_{26}\phi_{,22} + \zeta_{26}\phi^*_{,22} = -\bar{\rho}\,\omega^2 u_1,$$

$$e_{26}u_{1,12} - \xi_{22}\phi_{,22} - A_{22}\phi^*_{,22} = 0,$$

$$\zeta_{26}u_{1,22} - A_{22}\phi_{,22} - \xi^*_{22}\phi^*_{,22} = 0. \tag{53}$$

The solution, satisfying differential system (53) and boundary condition (52a) and (52b) is given by

$$u_1 = A \sin \eta x_2, \tag{54}$$

$$\phi = A\left(\frac{e_{26}\xi^*_{22} - A_{22}\zeta_{26}}{\xi_{22}\xi^*_{22} - A^2_{22}}\right)\left(\sin \eta x_2 - \frac{x_2}{b} \sin \eta b\right) + \frac{\phi^0}{b}x_2, \tag{55}$$

$$\phi^* = A\left(\frac{e_{26}A_{22} - \xi_{22}\zeta_{26}}{A^2_{22} - \xi_{22}\xi^*_{22}}\right)\left(\sin \eta x_2 - \frac{x_2}{b} \sin \eta b\right) + \frac{\phi^0}{b}x_2, \tag{56}$$

where

$$\eta^2 = \frac{\bar{\rho}\omega^2}{\hat{c}_{66}}. \tag{57}$$

Making use of Eqs. (54)–(56) in Eq. (52a), we get

$$A[\eta b \cos \eta b - k^2_{26} \sin \eta b - R'\eta^2 b^2 \sin \eta b] = -\left(\frac{e_{26} + \zeta_{26}}{\hat{c}_{66}}\right)\phi^0,$$

where

$$k^2_{26} = \frac{e^2_{26}\xi^*_{22} + \zeta^2_{26}\xi_{22} - 2A_{22}e_{26}\zeta_{26}}{(\xi_{22}\xi^*_{22} - A^2_{22})\hat{c}_{66}}.$$

Resonance occur when

$$\tan \eta b = \eta b/(k^2_{26} + R'\eta^2 b^2). \tag{58}$$

Following Bluestein and Tiersten [12], we have for $R' \ll 1$, $k^2_{26} \ll 1$

$$\omega^2 = \pi^2\hat{c}_{66}(1 - R' - 4k^2_{26}/\pi^2)^2/4\bar{\rho}b^2 \quad \text{(with electrode)} \tag{59}$$

and

$$\omega^2 = \pi^2\hat{c}_{66}/4\bar{\rho}b^2 \quad \text{(without electrode)}. \tag{60}$$

Hence, by equating Eqs. (45) and (47) to Eqs. (59) and (60), respectively, we have for $R' \ll 1$, $k^2_{26} \ll 1$

$$k^2_1 = \left(\frac{\pi^2}{12}\right)\frac{\hat{c}_{66}(1 + R' - 8k^2_{26}/\pi^2)}{c_{66}} \quad \text{(with electrode)} \tag{61}$$

$$k^2_1 = \frac{\pi^2}{12} \quad \text{(without electrode)}. \tag{62}$$

For thickness vibration in x_3 direction, following Ref. [47], the frequency is given by

$$\omega^2 = \pi^2 c_3(1 - R')^2/4\bar{\rho}b^2 \quad \text{(with electrode)}, \tag{63}$$

where

$$c_3 = \tfrac{1}{2}[c_{22} + c_{44} - \{(c_{22} - c_{44})^2 + 4c^2_{24}\}^{1/2}]$$

and

$$\omega^2 = \pi^2 c_3/4\bar{\rho}b^2 \quad \text{(without electrode)}. \tag{64}$$

Hence for $R' \ll 1$, $k_{26}^2 \ll 1$ on comparing Eqs. (49) and (50) with Eqs. (63) and (64), respectively, we have

$$k_3^2 = \left(\frac{\pi^2}{12}\right)\frac{(1+R')c_3}{c'_{44}} \quad \text{(with electrode)}, \tag{65}$$

$$k_3^2 = \left(\frac{\pi^2}{12}\right)\frac{c_3}{c'_{44}} \quad \text{(without electrode)}. \tag{66}$$

5. Numerical results and discussion

The thickness-shear resonance frequencies are computed numerically for a particular model of PZT. Following Deu and Benjeddou [32], the elastic, piezoelectric, dielectric and other dynamical coefficients are given in Table 1.

The variation of two-dimensional thickness shear resonance frequency in x_1 and x_3 directions, obtained from Eqs. (45), (47) and (49), (50), with the thickness of plate is shown in the Fig. 2(a) and (b), respectively. The solid and dotted curves correspond to the case when electrodes are present (EP) or absent (EA) over the faces of plate, respectively. The thickness-shear resonance frequency in x_1 as well as x_3 direction decreases with increase in the thickness of the plate. Thus the frequency of the thickness-shear mode can be controlled by the thickness dimension. It is also observed that the thickness shear resonance frequency in x_1 direction decreases when thin electrode film is deposit on the faces of plate while the effects of electrode film is almost negligible on the frequency of thickness mode along x_3 direction. It is clear from Eqs. (49) and (50) that when R' is very small, ω remains unchanged effectively in the cases of with and without electrode. The role of piezoelectric and dielectric constants does not come into picture in the x_3 direction while these affect in case of x_1 direction.

Figs. 3(a) and 4(b) depicts the variation of three-dimensional thickness shear resonance frequency in x_1 and x_3 directions obtained from Eqs. (59), (60) and (63), (64) with the thickness of plate. Comparison of Figs. 2 and 3 shows that

Table 1
Elastic, piezoelectric, dielectric and dynamical coefficients for PZT ceramic.

Elastic constants (GPa)	Piezoelectric constants (C/m^2)	Dielectric constants (nC/Vm)	Dynamical coefficients
$c_{22} = 126$	$e_{26} = 17$	$\xi_{22} = 15.3$	$\rho_{11}(\text{Kg/m}^3) = 7370$
$c_{24} = 38$	$\zeta_{26} = 19$	$\xi^*_{22} = 13.8$	$\rho_{12}(\text{Kg/m}^3) = -70$
$c_{44} = 20$		$a_{22} = 17.1$	$\rho_{22}(\text{Kg/m}^3) = 270$
$c_{66} = 23$			$\rho(\text{Kg/m}^3) = 7300$
			$\rho'(\text{Kg/m}^3) = 5700$
			$b'(m) = .0001$

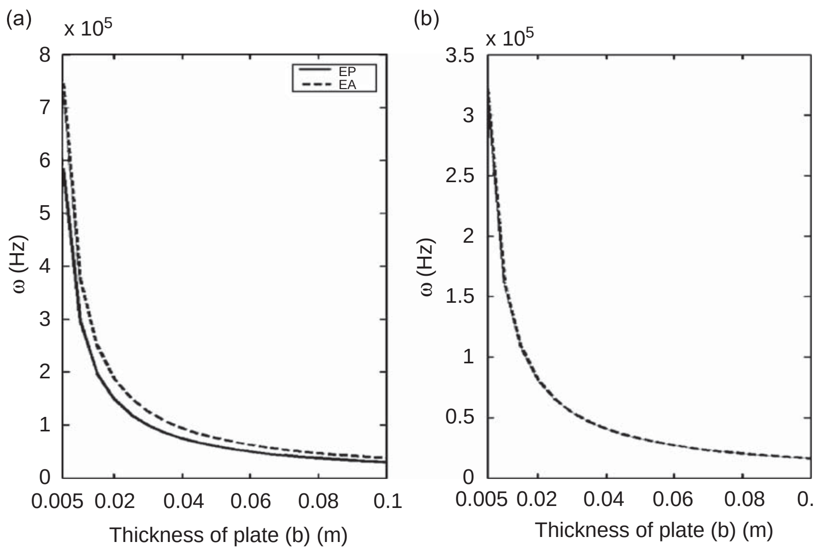

Fig. 2. Variation of two-dimensional thickness shear resonance frequencies (ω) with the thickness of plate: (a) in x_1 direction and (b) in x_3 direction; solid curves (electrodes are present (EP)) and dotted curves (electrodes are absent (EA)).

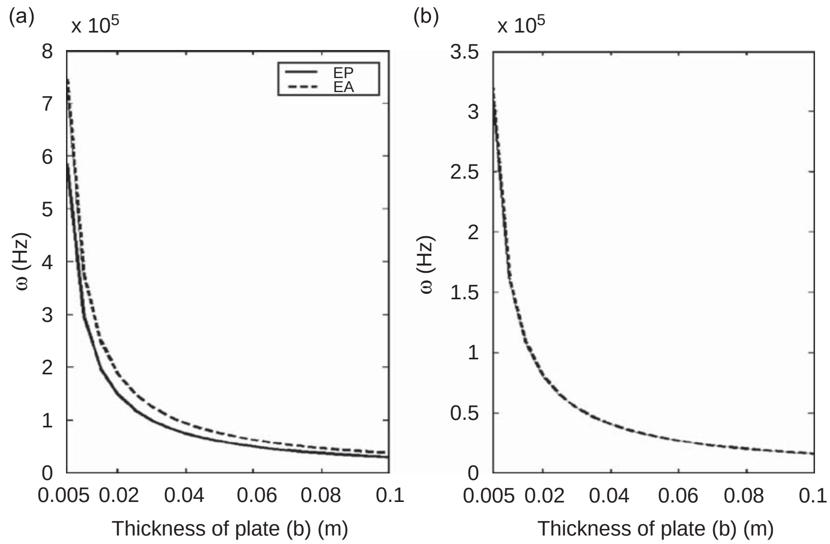

Fig. 3. Variation of three-dimensional thickness shear resonance frequencies (ω) with the thickness of plate: (a) in x_1 direction and (b) in x_3 direction solid curves (electrodes are present) and dotted curves (electrodes are absent).

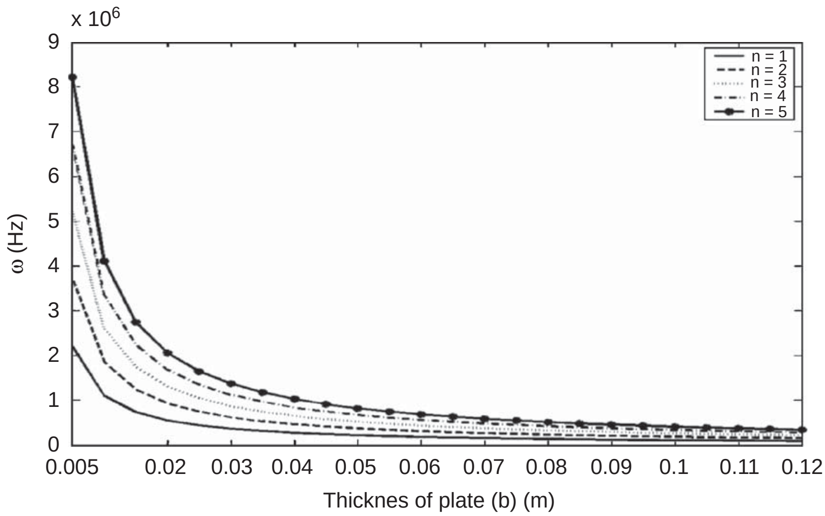

Fig. 4. Variation of three-dimensional thickness shear resonance frequency (ω) in x_1 direction with thickness of plate for first five modes.

the thickness-shear resonance frequency in x_1, x_3 directions obtained from the two-dimensional case and three-dimensional case is same.

Fig. 4 depicts the variation of thickness-shear resonance frequency obtained from the three-dimensional case i.e. solution of Eqs. (57) and (58) with the thickness of the plate for first five modes. It is observed that the thickness shear resonance frequency increases for the second mode in comparison to first mode and likewise for other modes of vibration.

6. Conclusion

In present paper, the constitutive equations and equations of motion for anisotropic porous piezoelectric materials are derived using variational principle. Further these equations are obtained for the two-dimensional case by expanding mechanical displacement and electrical potential function as power series. The constitutive equations are modified after truncation, adjustment of some terms and by introducing thickness-shear correction factors. The thickness-shear correction factors are determined for the monoclinic (2) crystal by comparing thickness-shear frequencies corresponding to the two and three-dimensional case. The resonance frequencies in thickness mode for two and three-dimensional case are obtained both analytically and numerically for a particular model of PZT. The thickness shear resonance frequency decreases with the thickness of plate. The electrode film deposited over the faces of the plate decreases the thickness shear resonance frequency in x_1 direction while effect on the thickness shear resonance frequency in x_3 direction is almost negligible.

Appendix A

$$c_{11}^{(0)} = c_{11} - \frac{c_{12}c_{12}}{c_{22}}, \quad c_{13}^{(0)} = c_{13} - \frac{c_{12}c_{23}}{c_{22}}, \quad c_{14}^{(0)} = k_3\left(c_{14} - \frac{c_{12}c_{24}}{c_{22}}\right), \quad c_{33}^{(0)} = c_{33} - \frac{c_{32}c_{23}}{c_{22}},$$

$$c_{34}^{(0)} = k_3\left(c_{43} - \frac{c_{42}c_{23}}{c_{22}}\right), \quad c_{44}^{(0)} = k_3^2\left(c_{44} - \frac{c_{42}c_{24}}{c_{22}}\right), \quad c_{55}^{(0)} = c_{55}, \quad c_{56}^{(0)} = k_1 c_{56},$$

$$c_{66}^{(0)} = k_1^2 c_{66}, \quad e_{11}^{(0)} = e_{11} - \frac{e_{12}c_{12}}{c_{22}}, \quad e_{13}^{(0)} = e_{13} - \frac{e_{12}c_{32}}{c_{22}}, \quad e_{14}^{(0)} = k_3\left(e_{14} - \frac{e_{12}c_{42}}{c_{22}}\right),$$

$$e_{25}^{(0)} = e_{25}, \quad e_{26}^{(0)} = k_1 e_{26}, \quad e_{35}^{(0)} = e_{35}, \quad e_{36}^{(0)} = k_1 e_{36}, \quad \zeta_{11}^{(0)} = \zeta_{11} - \frac{\zeta_{12}c_{12}}{c_{22}},$$

$$\zeta_{13}^{(0)} = \zeta_{13} - \frac{\zeta_{12}c_{32}}{c_{22}}, \quad \zeta_{14}^{(0)} = k_3\left(\zeta_{14} - \frac{\zeta_{12}c_{42}}{c_{22}}\right), \quad \zeta_{25}^{(0)} = \zeta_{25}, \quad \zeta_{26}^{(0)} = k_1\zeta_{26},$$

$$\zeta_{35}^{(0)} = \zeta_{35}, \quad \zeta_{36}^{(0)} = k_1\zeta_{36}, \quad m_{11}^{(0)} = m_{11} - \frac{m_{22}c_{21}}{c_{22}}, \quad m_{32}^{(0)} = k_3\left(m_{32} - \frac{m_{22}c_{24}}{c_{22}}\right),$$

$$m_{33}^{(0)} = m_{33} - \frac{m_{22}c_{23}}{c_{22}}, \quad \bar{e}_1^* = e_1^* - \frac{\zeta_{12}m_{22}}{c_{22}}, \quad \bar{\tilde{\zeta}}_1 = \tilde{\zeta}_1 - \frac{m_{22}e_{12}}{c_{22}}, \quad \bar{R} = R - \frac{m_{22}^2}{c_{22}},$$

$$\bar{\xi}_{11} = \xi_{11} + \frac{e_{12}^2}{c_{22}}, \quad \bar{\xi}_{22} = \xi_{22}, \quad \bar{\xi}_{23} = \xi_{23}, \quad \bar{\xi}_{33} = \xi_{33}, \quad \bar{\xi}_{11}^* = \xi_{11}^* + \frac{\zeta_{12}^2}{c_{22}}, \quad \bar{\xi}_{22}^* = \xi_{22}^*,$$

$$\bar{\xi}_{23}^* = \xi_{23}^*, \quad \bar{\xi}_{33}^* = \xi_{33}^*, \quad \bar{A}_{11} = A_{11} + \frac{e_{12}\zeta_{12}}{c_{22}}, \quad \bar{A}_{22} = A_{22}, \quad \bar{A}_{23} = A_{23}, \quad \bar{A}_{33} = A_{33},$$

$$\gamma_{11}^{(1)} = \frac{s_{3333}}{s_{1111}s_{3333} - s_{1133}^2}, \quad \gamma_{13}^{(1)} = \frac{s_{1133}}{s_{1111}s_{3333} - s_{1133}^2}, \quad \gamma_{33}^{(1)} = \frac{s_{1111}}{s_{1111}s_{3333} - s_{1133}^2}, \quad \gamma_{55}^{(1)} = \frac{1}{s_{1313}},$$

$$e_{11}^{(1)} = \gamma_{11}^{(1)}e_{11}^I + \gamma_{13}^{(1)}e_{13}^I, \quad e_{13}^{(1)} = \gamma_{31}^{(1)}e_{11}^I + \gamma_{33}^{(1)}e_{13}^I, \quad e_{35}^{(1)} = \gamma_{55}^{(1)}e_{35}^I + \gamma_{55}^{(1)}e_{35}^I,$$

$$\zeta_{11}^{(1)} = \gamma_{11}^{(1)}\zeta_{11}^I + \gamma_{13}^{(1)}\zeta_{13}^I, \quad \zeta_{13}^{(1)} = \gamma_{31}^{(1)}\zeta_{11}^I + \gamma_{33}^{(1)}\zeta_{13}^I, \quad \zeta_{35}^{(1)} = \gamma_{55}^{(1)}\zeta_{13}^I + \gamma_{55}^{(1)}\zeta_{31}^I,$$

$$m_{11}^{(1)} = \gamma_{11}^{(1)}m_{11}^I + \gamma_{13}^{(1)}m_{33}^I, \quad m_{33}^{(1)} = \gamma_{31}^{(1)}m_{11}^I + \gamma_{33}^{(1)}m_{33}^I, \quad m_{13}^{(1)} = \gamma_{55}^{(1)}m_{13}^I + \gamma_{55}^{(1)}m_{31}^I.$$

References

[1] W.G. Cady, *Piezoelectricity*, McGraw-Hill, New York, 1946.
[2] W.P. Mason, *Piezoelectric Crystals and Their Application to Ultrasonics*, Van Nostrand, Princeton, NJ, 1950.
[3] R. Holland, E.P. Eer Nisse, *Design of Resonant Piezoelectric Devices*, MIT Press, Cambridge and London, 1969.
[4] R.D. Mindlin, Forced thickness-shear and flexural vibrations of piezoelectric crystal plates, *Journal of Applied Physics* 23 (1952) 83–88.
[5] H.F. Tiersten, R.D. Mindlin, Forced vibrations of piezoelectric crystal plates, *Quarterly Applied Mathematics* 22 (1962) 107–119.
[6] H.F. Tiersten, *Linear Piezoelectric Plate Vibrations*, Plenum Press, New York, 1969.
[7] R.D. Mindlin, High frequency vibrations of piezoelectric crystal plates, *International Journal of Solids and Structures* 8 (1972) 895–906.
[8] R.D. Mindlin, *An introduction to the mathematical theory of vibrations of elastic plates*, US Army Signal Corps Engineering Laboratories, Fort Monmouth, NJ, 1955.
[9] A.G. Shaw, On the resonant vibrations of thick barium titanate disks, *Journal of the Acoustic Society of the America* 28 (1956) 38–50.
[10] J.L. Bleustein, Thickness-twist and face shear vibrations of a contoured crystal plate, *International Journal of Solids and Structures* 2 (1966) 351–360.
[11] R.D. Mindlin, P.C.Y. Lee, Thickness-shear and flexural vibrations of partially plated crystal plates, *International Journal of Solids and Structures* 2 (1966) 125–139.
[12] J.L. Bleustein, H.F. Tiersten, Forced thickness-shear vibrations of discontinuously plated piezoelectric crystal plates, *Journal of the Acoustic Society of the America* 43 (1968) 1311–1318.
[13] P.C.Y. Lee, J. Wang, Piezoelectrically forced thickness-shear and flexural vibrations of contoured quartz resonators, *Journal of Applied Physics* 79 (1996) 3411–3422.
[14] J. Wang, J.D. Yu, Y.K. Yong, T. Imai, A new theory for electroded piezoelectric plates and its finite element application for the forced vibrations of quartz crystal resonators, *International Journal of Solids and Structures* 37 (2000) 5653–5673.

[15] W.Q. Chen, Vibration theory of non-homogeneous spherically isotropic piezoelastic bodies, *Journal of Sound and Vibration* 236 (2000) 833–860.
[16] V.M. Bazhenov, Longitudinal vibration of piezoceramic rods, *Journal of Mathematical Sciences* 103 (2001) 240–244.
[17] J.C. Piquette, Quasistatic coupling coefficients for electrostrictive ceramics, *Journal of the Acoustic Society of the America* 111 (2001) 197–207.
[18] J.S. Yang, Shear horizontal vibrations of piezoelectric/ferroelectric wedge, *Acta Mechanica* 173 (2004) 13–17.
[19] E. Lioubimova, P. Schiavone, Steady-state vibrations of an unbounded linear piezoelectric medium, *Zeitschrift fur Angewandte Mathematik und Physik ZAMP* 57 (2006) 862–874.
[20] S.Y. He, W.S. Chen, Z.L. Chen, A uniformizing method for the free vibration analysis of metal–piezoceramic composite thin plates, *Journal of Sound and Vibration* 217 (1998) 261–281.
[21] H. Ding, R. Xu, Y. Chi, W. Chen, Free axisymmetric vibrations of transversely isotropic piezoelectric circular plates, *International Journal of Solids and Structures* 36 (1999) 4629–4652.
[22] H. Ding, R.Q. Xu, W. Chen, Exact solutions for free vibration of transversely isotropic piezoelectric circular plates, *Acta Mechanica Sinica* 16 (2000) 142–147.
[23] H.J. Ding, W.Q. Chen, R.Q. Xu, On the bending, vibration and stability of laminated rectangular plates with transversely isotropic layers, *Applied Mathematics and Mechanics* 22 (2001) 17–24.
[24] Y. Wang, R.Q. Xu, H.J. Ding, Free vibration of piezoelectric annular plate, *Journal of Zhejiang University Science* 4 (2003) 379–387.
[25] V.L. Karlash, Resonant electromechanical vibrations of piezoelectric plates, *International Applied Mechanics* 41 (2005) 709–747.
[26] W.Q. Chen, H.J. Ding, On free vibration of a functionally graded piezoelectric rectangular plate, *Acta Mechanica* 153 (2002) 207–216.
[27] W.Q. Chen, Z.G. Bian, C.F. Lv, H.J. Ding, 3D free vibration analysis of a functionally graded piezoelectric hollow cylinder filled with compressible fluid, *International Journal of Solids and Structures* 41 (2004) 947–964.
[28] S.M. Yang, J.W. Chiu, Smart structures-vibration of composites with piezoelectric materials, *Composites Structure* 25 (1993) 381–386.
[29] P.R. Heyliger, G. Ramirez, Free vibration of laminated circular piezoelectric plates and discs, *Journal of Sound and Vibration* 229 (2000) 935–956.
[30] J.A. Hernandes, S.F.M. Almeida, A. Naborrete, Stiffening effects on the free vibration behaviour of composite plates with PZT actuators, *Composite Structures* 49 (2000) 55–63.
[31] S.V. Senthil, R.C. Mewer, R.C. Batra, Analytical solution for the cylindrical bending vibration of piezoelectric composite plates, *International Journal of Solids and Structures* 41 (2004) 1625–1643.
[32] J.F. Deu, A. Benjeddou, Free vibration analysis of laminated plates with embedded shear-mode piezoceramic layers, *International Journal of Solids and Structures* 42 (2005) 2059–2088.
[33] X. Shu, Free vibration of laminated piezoelectric composite plates based on an accurate theory, *Composite Structures* 67 (2005) 375–382.
[34] G. Qing, J. Qiu, Y. Liu, Free vibration analysis of stiffened laminated plates, *International Journal of Solids and Structures* 43 (2006) 1357–1371.
[35] P. Topdar, A.H. Sheikh, N. Dhang, Vibration characteristics of composite/sandwich laminates with piezoelectric layers using a refined hybrid plate model, *International Journal of Mechanical Sciences* 49 (2007) 1193–1203.
[36] R.D. Mindlin, Equations of high frequency vibrations of thermopiezoelectric crystal plates, *International Journal of Solids and Structures* 10 (1974) 625–637.
[37] K. Xu, A.K. Noor, Y.Y. Tang, Three dimensional solutions for free vibrations of initially stressed thermoelectroelastic multilayered plates, *Computer Methods in Applied Mechanics and Engineering* 141 (1997) 125–139.
[38] S.K. Parashar, U.V. Wagner, Nonlinear longitudinal vibrations of transversely polarized piezoceramics: experiments and modeling, *Nonlinear Dynamics* 37 (2004) 51–73.
[39] S.K. Parashar, A. Dasgupta, U.V. Wagner, P. Hagedorn, Non-linear shear vibrations of piezoceramic actuators, *International Journal of Non-linear Mechanics* 40 (2005) 429–443.
[40] X.L. Huang, H.S. Shen, Nonlinear free and forced vibration of simply supported shear deformable laminated plates with piezoelectric actuators, *International Journal of Mechanical Sciences* 47 (2005) 187–208.
[41] L.P. Khoroshun, T.I. Dorodnykh, Piezoelectrics of polycrystalline structures, *Soviet Applied Mechanics* 27 (1991) 660–665.
[42] L.P. Khoroshun, P.V. Leshchenko, T.I. Dorodnykh, Effective electroelastic properties of polycrystals, *International Applied Mechanics* 30 (1994) 311–319.
[43] L.P. Khoroshun, T.I. Dorodnykh, the effective elastic constants of porous polycrystals of trigonal symmetry, *International Applied Mechanics* 37 (2001) 1290–1303.
[44] L.P. Khoroshun, T.I. Dorodnykh, Effective electrostrictive properties of stochastic two-component materials, *International Applied Mechanics* 44 (2008) 955–962.
[45] R.K. Gupta, T.A. Venkatesh, Electromechanical response of porous piezoelectric materials, *Acta Materialia* 54 (2006) 4063–4078.
[46] A.L. Cauchy, Sur l'equilibre et le movement d'une plaque don't l'elasticite n'est pas la meme dans tous les sens, *Exercise de Mathematique* 4 (1829) 1–14.
[47] R.D. Mindlin, High frequency vibrations of plated, crystal plates, *Progress in Applied Mechanics*, Praeger Anniversary Volume (1963) 73–84.